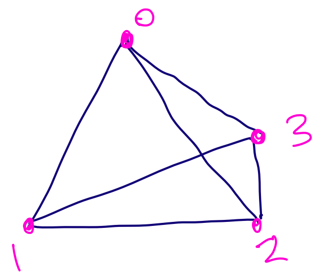
Goal : Techniques to study the homotopy type of various simplicial complexes.

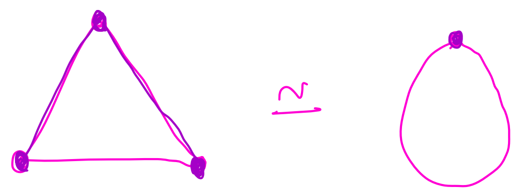
(I) (k-1)-skeleton of an n-simplex

Claim : $\simeq \vee S^{k-1}$



n-simplex : Vertices $\leftrightarrow \{0,1,\dots,n\}$
simplices $\leftrightarrow$ size $\leq k$ subsets

Lemma : $A \subset X$ subcplex, $A \simeq *$
$\Rightarrow X \simeq X/A$



In our case : let A = subcomplex of all subsets containing 0 + all subsets of size $\leq k-1$.

Is a cone w/ cone pt 0.

X/A : $\vee S^{k-1}$, a $(k-1)$-simplex for every $\{n_1,\dots,n_k\} \not\ni 0$.

(II) S: set, infinite
$T(S)^{\leq k}$: vertices $\leftrightarrow$ elts of S
simplices $\leftrightarrow$ size $\leq k$ subsets

$\simeq \vee S^{k-1}$.

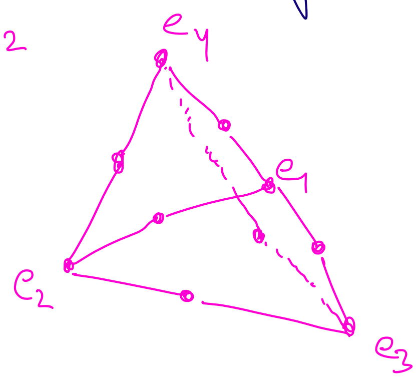
The same idea works. Pick A = subcomplex of all subsets containing s_0 + all size $\leq k-1$

III V : vector space

$T(V)^{\le k}$. Vertices $\leftrightarrow$ subspaces of V of $\dim \le k < \dim V$

Simplices $\leftrightarrow$ flags $V_0 \subset V_1 \subset \dots \subset V_p$

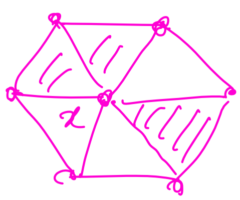
Eg: $V = \mathbb{Q}^4$ $\quad T(V)^{\le 2}$



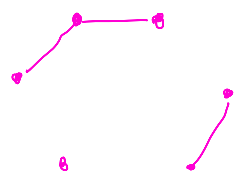
Claim : $T(V) \simeq \vee S^{k-1}$

Defn: Link of a vertex. $x \in \Delta^0$

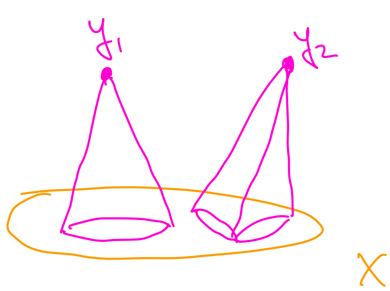
$$Lk_\Delta x = \{ \text{simplices } \tau \mid x \cup \tau \in \Delta,\ x \notin \tau \}$$



$lk\ x$:



Lemma : Suppose Y built from X by coning off subcomplexes of X (i.e. lk of every vertex $y \in Y \setminus X$ is in X)



suppose $X \simeq *$

Then $Y \simeq \bigvee_{y \text{ vertex of } Y \setminus X} (\underbrace{\Sigma\, Lk(y)}_{\text{suspension}})$

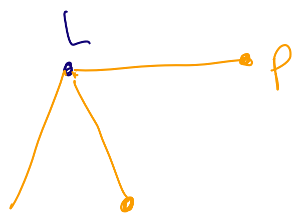
In particular, if all $Lk(y) \simeq *$, then $Y \simeq *$

if all $Lk(y) \simeq \vee S^d$, then $Y \simeq \vee S^{d+1}$

<u>The Proof</u>: Note $T(V)^{\leq 1}$: discrete set $\simeq \vee S^0$

Fix a line $L \subset V$.

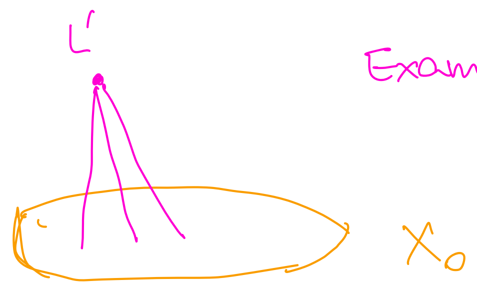
Take X_0 = subcplex of all ~~vector~~ subspaces containing L.



cone w/ cone pt L

$\simeq *$

X_1 : X_0, plus lines $L' \not\supset L$



Examine $Lk_X(L')$.

$L' \subset W$

$L \subset W$

So $L' + L \subset W$

$Lk_{X_0}(L')$: cone w/ cone pt $L' + L$

$\boxed{X_1 \simeq *}$

X_2 : X_1, plus planes $P \not\supset L$

$Lk_{X_1}(P)$

- lines $L' \subset P$
- $P \subset W$, $L \subset W$

$L' \subset P + L \subset W$

$Lk_{X_1}(P)$: Cone w/ cone pt $P+L$

$\boxed{X_2 \simeq *}$

X_{k-1} : X_{k-2} plus dim $k-1$ $U \not\supset L$

$\simeq *$

X_k : X_{k-1} plus dim k subspaces $U \not\supset L$

Note : $X_k = T(V)^{\leq k}$

$Lk_{X_{k-1}}(U)$: all $W \subset U$

note : no $W \supset U$ exist, since all such will have dim $\geq k+1$

so $Lk_{X_{k-1}}(U) = T(U)^{\leq k-1}$

$\simeq \vee S^{k-2}$

$\boxed{X_k \simeq \vee S^{k-1}}$

When $\dim V = n$, $T(V) := T(V)^{\leq n-1}$ "Solomon-Tits building"

$\simeq \vee S^{n-2}$

$\tilde{H}_{n-2}(T(V)) =: \underline{St(V)}$

Steinberg module

$GL(V) \curvearrowright St(V)$

$$\boxed{H_*(GL(V); St(V)) \cong H^{\nu-*}(GL(V); \mathbb{Q})}$$

~~IV~~ A Resolution of $St(V)$

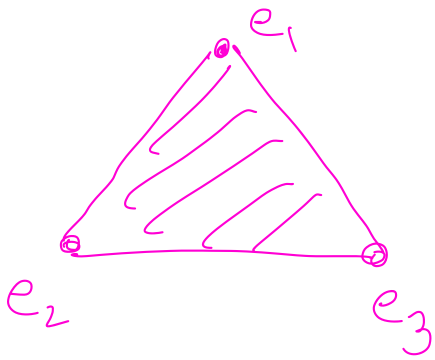
$K(V)$: vertices $\leftrightarrow$ vectors of V

simplices $\leftrightarrow$ all finite sets of vectors

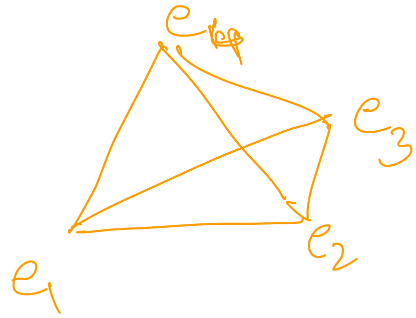
$K(V) \simeq *$

$L(V) \subset K(V)$: simplices $\leftrightarrow$ non-spanning finite sets of vectors

$V = \langle e_1, e_2, e_3, e_4 \rangle$

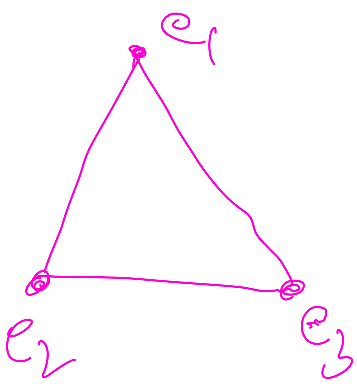


$L(V)$

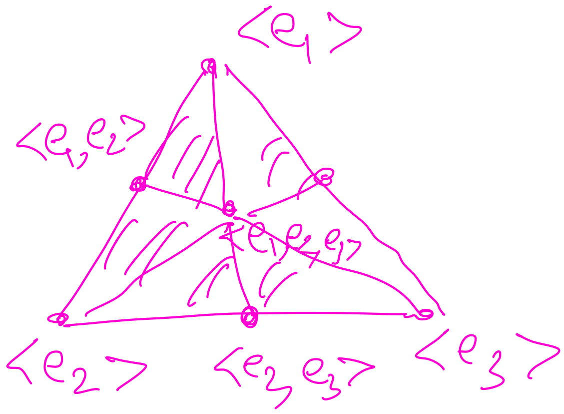


$K(V) \setminus L(V)$

There is a htpy equiv $L(V) \to T(V)$



$L(V)$



So: $K(V) \simeq *$, $L(V) \simeq T(V)$

$$\Rightarrow K(V)/L(V) \simeq \Sigma(T(V)) \simeq \vee S^{n-1}$$

chain complex of $K(V)/L(V)$:

$$\cdots \to \frac{C_m(K(V))}{C_m(L(V))} \to \frac{C_{m-1}(K(V))}{C_{m-1}(L(V))} \to \cdots$$

has H_* only in degree $n-1$.

Also: $\dfrac{C_m(K(V))}{C_m(L(V))} = 0$ for $m \leq n-2$

We get:

$$\cdots \longrightarrow \frac{C_n(K(V))}{C_n(L(V))} \longrightarrow \frac{C_{n-1}(K(V))}{C_{n-1}(L(V))} \longrightarrow St(V) \longrightarrow 0$$

A resolution of $St(V)$!